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Radiating Levi-Civita metric

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Abstract. A generalization of Levi–Civita's static solution for the exterior gravitational field of a cylinder of infinite length and finite cross section, corresponding to the field equations $T_{ab} = qk_ak_b$, $k_ak^a = 0$, is discussed. The expression for q is of the form $f'(u)/\rho$, which shows the typical cylindrical fall-off over the null hypersurface u = constant. It is pointed out that the passage of an outgoing wave affects permanently the static Levi–Civita space–time and the energy for this class of solutions is a particular case of the C-energy defined by Thorne. The geodesic equations for a test particle moving in the radial direction display a gravitational 'induction field' which is associated with a changing mass in the Newtonian field and is directed towards the axis of the cylinder. In contrast with the spherically symmetric case the induction field always acts to decrease the energy of a test particle on which it acts. It is shown that the mass per unit length of the static cylinder strongly affects the shear and divergence of the null congruence. The asymptotic behaviour of the Weyl tensor is analysed and a peeling theorem proved for this case.

1. Introduction to the metric

It is convenient for the discussion of cylindrically symmetric space–times with one degree of freedom in general relativity to take as exterior line-element the form given by Einstein and Rosen (1937), namely,

$$ds^2 = \exp\{2(\gamma-\psi)\}(dt^2-d\rho^2)-\exp(2\psi)\,dz^2-\rho^2\exp(-2\psi)\,d\varphi^2 \tag{1}$$

where γ and ψ are functions of ρ and t only. The author (1964, 1970) has shown that starting from any empty space solution of the gravitational field equations for a cylindrically symmetric field given by (1), it is always possible to construct a class of solutions to the non-empty space equations for a stress energy tensor

$$T_{ab} = qk_ak_b \tag{2}$$

where k_a is the null propagation vector for 'pure radiation' (Stachel 1969). Explicitly, if (ψ, γ) is any pair of functions in (1) satisfying the empty space field equations, the general solution of the field equations (2) for the same metric is given by $\{\psi, \gamma+f(u)\}$, f being arbitrary and $u = t-\rho$. Therefore, for any given empty space solution of the Einstein and Rosen metric one gets a class of solutions satisfying the pure radiation field equations (2) and q is of the form $f'(u)/\rho$. This shows the typical $1/\rho$ cylindrical fall-off over a null hypersurface u = constant and we identify ρ as the luminosity distance.

Applying this result to Levi–Civita's static solution and writing u as a coordinate, the metric field surrounding a *radiating* cylinder of infinite length and finite cross section is given by the metric

$$ds^2 = \exp\{2(f+\gamma-\psi)\}(du^2+2\,du\,d\rho)-\exp(2\psi)\,dz^2-\rho^2\exp(-2\psi)\,d\varphi^2 \tag{3}$$

with

$$\psi = -(1-c)\ln a - c(1-c)^{-1}\ln\rho$$
$$\gamma = -(1-2c)\ln a + c^2(1-c)^{-2}\ln\rho$$

where a and c are constants. In the earlier work by Levi–Civita (1919), Wilson (1920) and Marder (1958) it was argued that, for small c, the mass per unit length of the static cylinder is given by $M = c/2$ although as pointed out by Thorne (1965) this result is true only so long as the internal pressures of the cylinder are much smaller than its energy density.

The only nonvanishing component of the Ricci tensor for the metric (3) is given by

$$R_{00} = f'/\rho. \tag{4}$$

It will be shown in the next section that in order for the energy density of radiation to be positive, necessarily $f' \leqslant 0$.

2. The energy density

In this section we introduce a null tetrad with respect to which we later take the physical components of the Weyl tensor. We take k_a as one of the real null vectors, m_a as the other, and t_a as the complex null vector representing the two orthogonal space-like directions. The relations among the tetrad vectors are given by

$$k_a k^a = m_a m^a = t_a t^a = \bar{t}_a \bar{t}^a = t_a m^a = k_a t^a = 0$$
$$k_a m^a = t_a \bar{t}^a = 1.$$

Though the directions of the two real null vectors are uniquely fixed, an arbitrariness remains in their normalization. Following Stachel (1966) we give two choices:

Orthonormal tetrad — Null tetrads

$$\underset{0}{\mathrm{e}}{}^a = \mathrm{e}^{\psi-\gamma}\delta^a_t \qquad k^a = \underset{0}{\mathrm{e}}{}^a + \underset{1}{\mathrm{e}}{}^a, \qquad \tilde{k}^a = \mathrm{e}^{\psi-\gamma}k^a$$

$$\underset{1}{\mathrm{e}}{}^a = \mathrm{e}^{\psi-\gamma}\delta^a_\rho \qquad m^a = \tfrac{1}{2}(\underset{0}{\mathrm{e}}{}^a - \underset{1}{\mathrm{e}}{}^a), \qquad \tilde{m}^a = \mathrm{e}^{\gamma-\psi}m^a$$

$$\underset{2}{\mathrm{e}}{}^a = \mathrm{e}^{-\psi}\delta^a_z \qquad t^a = \frac{1}{\sqrt{2}}(\underset{2}{\mathrm{e}}{}^a + \mathrm{i}\underset{3}{\mathrm{e}}{}^a)$$

$$\underset{3}{\mathrm{e}}{}^a = (\mathrm{e}^{\psi}/\rho)\delta^a_\varphi \qquad \bar{t}^a = \frac{1}{\sqrt{2}}(\underset{2}{\mathrm{e}}{}^a - \mathrm{i}\underset{3}{\mathrm{e}}{}^a).$$

We find the null tetrad $(\tilde{k}^a, \tilde{m}^a, t^a, \bar{t}^a)$ more convenient for the discussion of the field equations since in this case k_a is the gradient of u. From the tetrad relations we notice that the only nonvanishing projection of equation (2) onto the null tetrad will be

$$q = T_{ab}m^a m^b = -(1/8\pi)R_{ab}m^a m^b = -(1/8\pi)f'/\rho.$$

Since q, being the energy density measured in this frame, must be positive, it follows that $f' \leqslant 0$.

By stipulating in (3) that (i), $f(u) = 0$ for $u \leqslant 0$, and (ii), $f(u) \to$ constant as $u \to \infty$, and following an argument exactly similar to the one given by Marder (1958) it can be shown that the passage of an outgoing wave produces a permanent change in Levi–Civita's exterior metric, and consequently in the source. Alternatively, it is easy to show that f contributes to what Stachel (1966) calls the disposable gravitational mass, the amount available for radiation. Also by analogy with gravitational news function he defines $f(u)$ as the pure radiation news function. The energy for this class of solutions is a particular case of the C-energy defined by Thorne (1965).

The effect of the static field on that of radiation can easily be seen from the optical scalars. Straightforward computation for the divergence and shear of the null congruence gives

$$\theta = k^a{}_{;a} = \exp\{2(\psi-\gamma-f)\}/\rho$$

$$|\sigma| = (\tfrac{1}{2}k_{(a;b)}k^{a;b}-\tfrac{1}{4}\theta^2)^{1/2} = \frac{(1+c)^2\theta^2}{4(1-c)^2}.$$

It is clear that the value of c strongly affects the shear (which vanishes only for $c = -1$) and divergence. It will be shown in § 4 that the presence of the static field makes many quantities fall off asymptotically more slowly than $1/\rho$, which is characteristic for a cylindrical radiation field.

3. Geodesics

We calculate the geodesics for the metric (3) in the plane z = constant. Since ϕ is a cyclic coordinate, the corresponding relative angular momentum per unit mass is always a constant and is given by

$$-\frac{\partial \mathscr{L}}{\partial \dot{\phi}} = \mathrm{e}^{-2\psi}\rho^2\dot{\phi} \equiv l.$$

The relative energy per unit mass is given by

$$\frac{\partial \mathscr{L}}{\partial \dot{u}} = \exp\{2(f+\gamma-\psi)\}(\dot{u}+\dot{\rho}) \equiv K.$$

It is to be noted that K is a constant only when f is. The geodesic equations themselves may be written:

$$\frac{\mathrm{d}l}{\mathrm{d}s} = 0 \tag{5}$$

$$\frac{\mathrm{d}K}{\mathrm{d}s} = (1+l^2\mathrm{e}^{2\psi}\rho^{-2})f' \tag{6}$$

$$\frac{\mathrm{d}\dot{\rho}}{\mathrm{d}s} = f'\dot{u}^2 - \frac{c\exp\{2(\psi-\gamma-f)\}}{(1-c)^2\rho} + \frac{(2c-1)l^2\exp\{2(2\psi-\gamma-f)\}}{(1-c)\rho^3}. \tag{7}$$

The difference between these equations and those for Levi–Civita's static case lie in terms containing $f'(u)$. By analogy with the spherically symmetric solution discussed by Lindquist *et al.* (1965) the first term on the right-hand side of (7) may be termed the 'induction field' associated with a changing mass in the Newtonian field (next term) and is directed towards the axis of the cylinder, since $f'(u) \leqslant 0$. But in contrast to the spherically symmetric case it always acts to decrease the energy of the test particle on which it acts (equation (6)). Of course, this is not surprising and is to be expected from the Newtonian analogue. The last term in (7) represents the centrifugal force.

4. Asymptotic form of the Weyl tensor

The only nonvanishing complex components of the Weyl tensor with respect to the null tetrad $(k^a, m^a, t^a, \bar{t}^a)$ are given by

$$\Psi_0 = -C_{abcd}k^a t^b k^c t^d = -\frac{c(1+c)\exp\{2(\psi-\gamma-f)\}}{(1-c)^3\rho^2}$$

$$\Psi_2 = -C_{abcd}\bar{t}^a m^b k^c t^d = -\frac{c\exp\{2(\psi-\gamma-f)\}}{(1-c)^2\rho^2}$$

$$\Psi_4 = -C_{abcd}\bar{t}^a m^b \bar{t}^c m^d = \tfrac{1}{4}\Psi_0 + \frac{(1+c)\exp\{2(\psi-\gamma-f)\}f'}{2(1-c)\rho}.$$

It may be noted that $2(\psi-\gamma)$ has its maximum value when $c = -1$; for this value $\exp(2\psi-2\gamma) = \rho^{1/2}$. Therefore, $\exp(2\psi-2\gamma)$ is always of order less than $\rho^{1/2}$ except when $c = -1$ (the shear-free case). Since the highest power occurring in the Weyl tensor is of order $\exp(2\psi-2\gamma)/\rho$, this means that the Weyl tensor always vanishes asymptotically. At any rate to order $\exp(2\psi-2\gamma)/\rho$, only $\Psi_4 \neq 0$ and the metric is of type N. To order $\exp(2\psi-2\gamma)/\rho^2$, Ψ_0, Ψ_2, Ψ_4 all the three components do not vanish and the metric is of type I.

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